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ON DUALITIES IN THE OSCILLATIONS OF NATURALLY CURVED AND PRETWISTED RODS

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Abstract—The pertinent equations of naturally curved and pretwisted rods, in the form of compatibility, equilibrium and constitutive relations are obtained under the assumptins of infinitesimal deformations and material isotropy. Then by forming the expressions for various energy terms, the equations of motion of the rod are obtained via Hamilton's principle and the complementary energy principle. On comparing these two forms of equations of motion, and the associated boundary conditions certain dualities are exposed. Finally the equations of some special rods, including the plane arch and the straight pretwisted rod, are examined.

INTRODUCTION

Governing equations of elastodynamic systems are often derived in the form of dynamic equilibrium equations. It is also well known that the equilibrium equations can be readily obtained via Hamilton's principle. It is not as well known that the governing equations can also take the form of compatibility equations obtained from complementary energy principle[1–3]. The advantages of one formulation over the other depend generally on the system to be analysed. In an approximate analysis simultaneous application of both formulations can yield a deeper insight into the characteristics of the system.

The availability of the two formulations also opens the way for new avenues of enquiry. Thus having obtained the governing equations of a system A in one formulation we may pose the question "is there a system B whose governing equations in the other formulation are identical to those obtained for A?". If the answer is in the affirmative than systems. A and B will have certain characteristics in common even though they may be physically different. We can then refer to A and B as dual systems.

In some recent papers[4–6] dualities in straight beams were investigated and extended to some simple frameworks. Our objective in this paper is to explore the dualities in the more general case of naturally curved and twisted rods. We shall first obtain the pertinent equations of such rods for linear oscillations. Then by invoking Hamilton's Principle and the complementary energy principle we shall expose the inherent dualities in the oscillation of such rods. To highlight the dual quantities we shall employ the vector notation.

1. GOVERNING EQUATIONS OF PRETWISTED-NATURALLY CURVED RODS

Consider a curved and pretwisted slender rod oscillating in space. To describe the motion of this rod we introduce an intertial frame of reference X, Y, Z and for convenience we also define the local coordinate axes x_0, y_0, z_0 and x, y, z as shown in Fig. 1. Detailed governing equations for small oscillations of such rods have been obtained by a number of investigators (see for instance[8–12]). Here we shall outline a brief derivation of these equations in order to expose an inherent duality in the oscillations of spatial rods. In writing the governing equations of the rod we shall consider only small deformations due to stretching, shearing, torsion and bending.

In the general case a further deformation due to the warping of the cross sections arises.‡

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‡For circular cross sections warping is absent.

However for slender rods this effect is small relative to the other deformations and in the present derivation we shall neglect it. It will be useful to bear in mind that the rate of charge of orientation of reference x, y, z relative to X, Y, Z can be represented by a vector $\boldsymbol{\kappa}(=\mathbf{i}\kappa_x + \mathbf{j}\kappa_y + \mathbf{t}\kappa_z)$ wherein $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{t}$ are the unit vectors along x, y, z axes, and κ_x and κ_y are components of curvature vector and κ_z is the measure of curve twist. Finally it should be noted that under the assumptions of infinitesimal deformation we may use $\mathbf{t}$ as the unit tangent vector to the centreline at A.

1.1 *Displacement strain relations*

Detailed derivation of strain displacement relations for curved and pretwisted rods can be found in a number of references (see for instance Washizu[13]). Here we give a concise derivation in vector notation. The position of any point of the rod can be described by the three components of displacement vector $\mathbf{u}$ and the three components of the rotation vector $\boldsymbol{\theta}$. The former will locate the origin of the local axes x, y, z while the latter will fix the orientation of these axes with respect to the axes x_0, y_0, z_0 (see Fig. 1). In terms of $\mathbf{u}$ and $\boldsymbol{\theta}$ we define a centreline strain vector $\mathbf{e}$ given by

$$\mathbf{e} = \frac{\partial \mathbf{u}}{\partial s} + \mathbf{t} \times \boldsymbol{\theta} = \mathbf{i}(\epsilon_{zx} - \theta_y) + \mathbf{j}(\epsilon_{zx} + \theta_x) + \mathbf{t}(\epsilon_{zz}). \tag{1}$$

Likewise we define a vector of curvature changes $\Delta\boldsymbol{\kappa}(=\boldsymbol{\kappa} - \boldsymbol{\kappa}_0)$, given by

$$\Delta\boldsymbol{\kappa} = \frac{\partial \boldsymbol{\theta}}{\partial s} = \mathbf{i}\Delta\kappa_x + \mathbf{j}\Delta\kappa_y + \mathbf{t}\Delta\kappa_z. \tag{2}$$

Now noting that

$$\frac{\partial}{\partial s}(\ ) = (\ )' + \boldsymbol{\kappa} \times (\ ), \tag{3}$$

where the prime denotes differentiation with respect to the local axes, we may express the derivatives in eqns (1) and (2) in terms of the local coordinate system. However before doing so we observe that under the assumptions of linear theory, it is permissible to replace $\boldsymbol{\kappa}$ in eqn (3) by $\boldsymbol{\kappa}_0$, i.e. the initial curvature vector. Hence eqns (1) and (2) may be rewritten as

$$\mathbf{e} = \mathbf{u}' + \boldsymbol{\kappa}_0 \times \mathbf{u} + \mathbf{t} \times \boldsymbol{\theta} \tag{4}$$

and

$$\Delta\boldsymbol{\kappa} = \boldsymbol{\theta}' + \boldsymbol{\kappa}_0 \times \boldsymbol{\theta}. \tag{5}$$

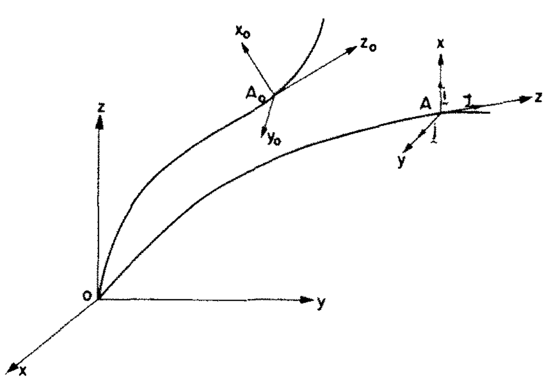

Fig. 1. The orthogonal coordinate systems of the rod in the undeformed and deformed states (z_0 and z are tangential to the centreline).

Equations (4) and (5) are the required displacement strain relations. Strain configurations calculated from these equations will implicitly satisfy the compatibility requirements providing of course $\mathbf{u}$ and $\boldsymbol{\theta}$ are continuous functions.

1.2 *The dynamical equations of equilibrium*

The translational and rotational equations of equilibrium of an infinitesimal element of the rod can be written as (see for example [8, 9])

$$\frac{\partial \mathbf{Q}}{\partial s} = \frac{\partial \mathbf{R}}{\partial t} \tag{6}$$

$$\frac{\partial \mathbf{M}}{\partial s} + \mathbf{t} \times \mathbf{Q} = \frac{\partial \boldsymbol{\Sigma}}{\partial t} \tag{7}$$

where $\mathbf{R}$ and $\boldsymbol{\Sigma}$ are the linear and angular momentum vectors.

Again using eqn (3) and noting that we may replace $\boldsymbol{\kappa}$ by $\boldsymbol{\kappa}_0$, we can express the above equilibrium equations in terms of local coordinate system, by doing so we get

$$\mathbf{Q}' + \boldsymbol{\kappa}_0 \times \mathbf{Q} = \dot{\mathbf{R}} \tag{8}$$

$$\mathbf{M}' + \boldsymbol{\kappa}_0 \times \mathbf{M} + \mathbf{t} \times \mathbf{Q} = \dot{\boldsymbol{\Sigma}}. \tag{9}$$

1.3 *The constitutive relations*

The constitutive relations for the rod, assuming the material to be linearly elastic and isotropic, can be written as follows,

$$\begin{aligned} Q_x &= \gamma GA(\epsilon_{zx} - \theta_y) \\ Q_y &= \gamma GA(\epsilon_{zy} + \theta_x) \\ Q_z &= AE\epsilon_{zz} \end{aligned} \tag{10}$$

$$\begin{aligned} M_x &= EI_x \Delta\kappa_x \\ M_y &= EI_y \Delta\kappa_y \\ M_z &= GI_z \Delta\kappa_z \end{aligned} \tag{11}$$

where G and E are the shear and Young moduli respectively, A is the cross sectional area, I_x, I_y and I_z are the second moments of area and γ is the shape factor for shear. We can write the above equations collectively by introducing the dyadics $\underline{\mathbf{GA}}(=\mathbf{ii}\gamma GA + \mathbf{jj}\gamma GA + \mathbf{tt}AE)$ and $\underline{\mathbf{EI}}(=\mathbf{ii}EI_{xx} + \mathbf{jj}EI_{yy} + \mathbf{tt}GI_z)$. Thus we have that

$$\mathbf{Q} = \underline{\mathbf{GA}}\mathbf{e} \tag{12}$$

and

$$\mathbf{M} = \underline{\mathbf{EI}}\Delta\boldsymbol{\kappa}. \tag{13}$$

Apart from the above static constitutive relations we have a set of dynamic constitutive relations. These are the momenta velocity relations and they may be expressed as

$$\mathbf{R} = \underline{\mathbf{N}}\dot{\mathbf{u}} \tag{14}$$

$$\boldsymbol{\Sigma} = \underline{\mathbf{J}}\dot{\boldsymbol{\theta}} \tag{15}$$

where $\underline{\mathbf{N}}(=\mathbf{ii}\mu A + \mathbf{jj}\mu A + \mathbf{tt}\mu A)$ is the mass dyadic and $\underline{\mathbf{J}}$ is the dyadic of moments of inertia defined as

$$\underline{\mathbf{J}} = \left(\int_A \mu y^2 \,\mathrm{d}A\right)\mathbf{ii} + \left(\int_A \mu x^2 \,\mathrm{d}A\right)\mathbf{jj} + \left(\int_A \mu (x^2+y^2)\,\mathrm{d}A\right)\mathbf{tt}$$

In these latter dyadics μ is the mass density of the rod.

2. VARIATIONAL DERIVATION OF EQUATIONS OF MOTION

In this section we derive the equations of motion of the beam via Hamilton's principle and the complementary energy principle. In the former case continuous generalised displacement functions are used as the competing admissible functions and the Euler-Lagrange equations emerge as the dynamic equilibrium equations. In the latter case the admissible functions are sets of equilibriating force quantities and the Euler–Lagrange equations emerge as the (dynamic) compatibility equations. On comparing the two forms of equations of motion we point out certain dualities in the oscillation characteristics of the beam.

2.1 *Hamiltonian formulation*

Hamilton's principle states that amongst all compatible strain fields the one that also satisfies the equilibrium equation is distinguished by the following variational statement

$$\delta \int_{t_1}^{t_2} (T^* - U)\,\mathrm{d}t = 0 \tag{16}$$

where T^* is the complementary kinetic energy and U is the strain energy of the system. In the case of the rod

$$\delta T^* = \int_0^L \mathbf{R} \cdot \delta \dot{\mathbf{u}}\,\mathrm{d}s + \int_0^L \mathbf{\Sigma} \cdot \delta \dot{\boldsymbol{\theta}}\,\mathrm{d}s \tag{17}$$

The first part of eqn (17) is concerned with the translational motion and the second part accounts for the rotational motion. The variation of strain energy is given by

$$\delta U = \int_0^L \mathbf{M} \cdot \delta \Delta \boldsymbol{\kappa}\,\mathrm{d}s + \int_0^L \mathbf{Q} \cdot \delta \mathbf{e}\,\mathrm{d}s. \tag{18}$$

The first part of eqn (18) expresses the variation of strain energy in bending and torsion while the second part is concerned with the energy in shear and stretching.

We may now express $\Delta\boldsymbol{\kappa}$ and $\mathbf{e}$ in terms of $\mathbf{u}$ and $\boldsymbol{\theta}$ through the strain-displacement eqns (4) and (5). In this way the compatibility requirements will be satisfied directly. Finally we may also express $\mathbf{R}, \mathbf{\Sigma}, \mathbf{M}$ and $\mathbf{Q}$ in terms of $\mathbf{u}$ and $\boldsymbol{\theta}$ through the constitutive eqns (12)–(15). Then eqn (16) may be expressed as follows

$$\delta \int_{t_1}^{t_2} \int_0^L \frac{1}{2}\{\dot{\mathbf{u}}\underline{\mathbf{N}}\dot{\mathbf{u}} + \dot{\boldsymbol{\theta}}\underline{\mathbf{J}}\dot{\boldsymbol{\theta}}) - ([\boldsymbol{\theta}' + \boldsymbol{\kappa}_0 \times \boldsymbol{\theta}]\underline{\mathbf{EI}}[\boldsymbol{\theta}' + \boldsymbol{\kappa}_0 \times \boldsymbol{\theta}] + [\mathbf{u}' + \boldsymbol{\kappa}_0 \times \mathbf{u} + \mathbf{t} \times \boldsymbol{\theta}]\underline{\mathbf{GA}}[\mathbf{u}' + \boldsymbol{\kappa}_0 \times \mathbf{u} + \mathbf{t} \times \boldsymbol{\theta}]\}\,\mathrm{d}s = 0 \tag{19}$$

In eqn (19) the only quantities subject to independent variations are the vectors $\mathbf{u}$ and $\boldsymbol{\theta}$. Carrying out the variation and the usual integrations by parts we obtain the following equations as conditions of stationarity of the functional in eqn (19).

Along $s = 0$ *to* $s = L$

$$-\underline{\mathbf{N}} \cdot \ddot{\mathbf{u}} + ([\mathbf{u}' + \boldsymbol{\kappa}_0 \times \mathbf{u} + \mathbf{t} \times \boldsymbol{\theta}] \cdot \underline{\mathbf{GA}})' - (\mathbf{u}' + \boldsymbol{\kappa}_0 \times \mathbf{u} + \mathbf{t} \times \boldsymbol{\theta}) \cdot \underline{\mathbf{GA}} \times \boldsymbol{\kappa}_0 = 0 \tag{20}$$

$$-\underline{\mathbf{J}}\ddot{\boldsymbol{\theta}} + ([\boldsymbol{\theta}' + \boldsymbol{\kappa}_0 \times \boldsymbol{\theta}]\underline{\mathbf{EI}})' - (\boldsymbol{\theta}' + \boldsymbol{\kappa}_0 \times \boldsymbol{\theta})\underline{\mathbf{EI}} \times \boldsymbol{\kappa}_0 - (\mathbf{u}' + \boldsymbol{\kappa}_0 \times \mathbf{u} + \mathbf{t} \times \boldsymbol{\theta})\underline{\mathbf{GA}} \times \mathbf{t} = 0 \tag{21}$$

At $s = 0, L$

$$[\mathbf{u}' + \boldsymbol{\kappa}_0 \times \mathbf{u} + \mathbf{t} \times \boldsymbol{\theta}] \cdot \underline{\mathbf{GA}} \cdot \delta \mathbf{u} = 0 \tag{22}$$

i.e. either $\mathbf{u}$ is prescribed or $\mathbf{Q} = 0$

$$[\boldsymbol{\theta}' + \boldsymbol{\kappa}_0 \times \boldsymbol{\theta}] \cdot \underline{\mathbf{EI}} \cdot \delta \boldsymbol{\theta} = 0$$

$$\text{i.e. either } \boldsymbol{\theta} \text{ is prescribed or } \mathbf{M} = 0. \tag{23}$$

Equations (20) and (21) are the dynamic equilibrium equations. They may be readily verified by comparing them with eqns (8) and (9) bearing in mind the constitutive relations (12)–(15). Likewise the interpretation of the natural boundary conditions can be verified through the use of constitutive eqns (13) and (12).

2.2 *Complementary energy formulation*

In this formulation it is convenient to work in terms of impulse rather than force variables [1–6]. To this end we integrate the equilibrium eqns (8) and (9) with respect to time and without loss of generality we take the constant of integration as zero. Then the balance equations of impulses and momenta may be written as

$$\mathbf{q}' \times \boldsymbol{\kappa}_0 \times \mathbf{q} = \mathbf{R} \tag{24}$$

$$\mathbf{m}' + \boldsymbol{\kappa}_0 \times \mathbf{m} + \mathbf{t} \times \mathbf{q} = \boldsymbol{\Sigma} \tag{25}$$

where $\mathbf{q}$ and $\mathbf{m}$ are impulses corresponding to $\mathbf{Q}$ and $\mathbf{M}$ respectively i.e. $\dot{\mathbf{q}} = \mathbf{Q}$ and $\dot{\mathbf{m}} = \mathbf{M}$.

Now the complementary energy principle states that amongst all impulse configurations that satisfy the equilibrium equations, the one that also satisfies the compatibility equations is distinguished by the following variational ststement

$$\delta \int_{t_1}^{t_2} (U^* - T)\, \mathrm{d}t = 0 \tag{26}$$

where U^* is the complementary strain energy and T is the kinetic energy of the system. For the case of the rod we have

$$\begin{aligned} \delta U^* &= \int_0^L \Delta\boldsymbol{\kappa} \cdot \delta \dot{\mathbf{m}}\, \mathrm{d}s + \int_0^L \mathbf{e} \cdot \delta \dot{\mathbf{q}}\, \mathrm{d}s \\ \delta T &= \int_0^L \dot{\mathbf{u}} \cdot \delta \mathbf{R}\, \mathrm{d}s + \int_0^L \dot{\boldsymbol{\theta}} \cdot \delta \boldsymbol{\Sigma}\, \mathrm{d}s \end{aligned} \tag{27}$$

This time we express $\mathbf{R}$ and $\boldsymbol{\Sigma}$ in terms of $\mathbf{q}$ and $\mathbf{m}$ via eqns (24) and (25) and in so doing we implicitly satisfy the equilibrium equations. We may also express $\Delta\boldsymbol{\kappa}$, $\mathbf{e}$, $\dot{\mathbf{u}}$ and $\dot{\boldsymbol{\theta}}$ in terms of $\mathbf{m}$ and $\mathbf{q}$ via the static and dynamic constitutive eqns (12)–(15). Then eqn (26) may be written as

$$\delta \int_t^t \int_0^L \frac{1}{2} \{(\dot{\mathbf{m}} \underline{\mathbf{EI}}^{-1} \dot{\mathbf{m}} + \dot{\mathbf{q}} \underline{\mathbf{GA}}^{-1} \dot{\mathbf{q}}) - ([\mathbf{q}' + \boldsymbol{\kappa}_0 \times \mathbf{q}] \underline{\mathbf{N}}^{-1}$$

$$[\mathbf{q}' + \boldsymbol{\kappa}_0 \times \mathbf{q}] + [\mathbf{m}' + \boldsymbol{\kappa}_0 \times \mathbf{m} + \mathbf{t} \times \mathbf{q}] \underline{\mathbf{J}}^{-1} [\mathbf{m}' + \boldsymbol{\kappa}_0 \times \mathbf{m} + \mathbf{t} \times \mathbf{q}]\}\, \mathrm{d}s = 0. \tag{28}$$

In eqn (28) the only quantities subject to variation are the impulse vectors $\mathbf{q}$ and $\mathbf{m}$. Carrying out the variation and the resulting integrations by parts, we obtain the following equations for the stationarity of the functional in eqn (28).

Along s = 0, *to* s = L

$$-\underline{\mathbf{EI}}^{-1} \ddot{\mathbf{m}} + ([\mathbf{m}' + \boldsymbol{\kappa}_0 \times \mathbf{m} + \mathbf{t} \times \mathbf{q}] \underline{\mathbf{J}}^{-1})' - [\mathbf{m}' + \boldsymbol{\kappa}_0 \times \mathbf{m} + \mathbf{t} \times \mathbf{q}] \underline{\mathbf{J}}^{-1} \times \boldsymbol{\kappa}_0 = 0 \tag{29}$$

$$-\underline{\mathbf{GA}}^{-1} \ddot{\mathbf{q}} + ([\mathbf{q}' + \boldsymbol{\kappa}_0 \times \mathbf{q}] \underline{\mathbf{N}}^{-1})' - [\mathbf{q}' + \boldsymbol{\kappa}_0 \times \mathbf{q}] \underline{\mathbf{N}}^{-1} \times \boldsymbol{\kappa}_0 - [\mathbf{m}' + \boldsymbol{\kappa}_0 \times \mathbf{m} + \mathbf{t} \times \mathbf{q}] \underline{\mathbf{J}}^{-1} \times \boldsymbol{\kappa}_0 = 0. \tag{30}$$

At s = 0, L

$$[\mathbf{m}' + \boldsymbol{\kappa}_0 \times \mathbf{m} + \mathbf{t} \times \mathbf{q}]\underline{\mathbf{J}}^{-1} \cdot \delta\mathbf{m} = 0$$

i.e. either **m** is prescribed or $\boldsymbol{\theta} = 0$ (31)

$$[\mathbf{q}' + \boldsymbol{\kappa}_0 \times \mathbf{q}]\underline{\mathbf{N}}^{-1} \cdot \delta\mathbf{q} = 0$$

i.e. either **q** is prescribed or $\mathbf{u} = 0$. (32)

Equations (29) and (30) are the dynamic compatibility equations. They may be readily verified by comparing them with eqns (4) and (5)—bearing in mind the constitutive eqns (12)–(15). The natural boundary conditions are again easily interpreted via the constitutive equations.

Now on comparing eqns (29) and (30) with eqns (20) and (21) we notice a remarkable duality. The following quantities are seen to be dual variables

$$\mathbf{q} \leftrightarrow \boldsymbol{\theta} \qquad \mathbf{m} \leftrightarrow \mathbf{u}$$
$$\underline{\mathbf{EI}} \leftrightarrow \underline{\mathbf{N}}^{-1} \qquad \underline{\mathbf{GA}} \leftrightarrow \underline{\mathbf{J}}^{-1}. \tag{33}$$

The boundary conditions also possess the same dual characteristics but it is of interest to note that the physical meaning of the dual boundary conditions can be different. Thus in the Hamiltonian Formulation the free end condition is described by

$$[\mathbf{u}' + \boldsymbol{\kappa}_0 \times \mathbf{u} + \mathbf{t} \times \boldsymbol{\theta}]\underline{\mathbf{GA}} = 0 \qquad \text{i.e. } \mathbf{Q} = 0$$
$$[\boldsymbol{\theta}' + \boldsymbol{\kappa}_0 \times \boldsymbol{\theta}]\underline{\mathbf{EI}} = 0 \qquad \text{i.e. } \mathbf{M} = 0$$

The dual boundary conditions in the complementary formulation belong to the fixed end viz

$$[\mathbf{m}' + \boldsymbol{\kappa}_0 \times \mathbf{m} + \mathbf{t} \times \mathbf{q}]\underline{\mathbf{J}}^{-1} = 0 \qquad \text{i.e. } \boldsymbol{\theta} = 0$$
$$[\mathbf{q}' + \boldsymbol{\kappa}_0 \times \mathbf{q}]\underline{\mathbf{N}}^{-1} = 0 \qquad \text{i.e. } \mathbf{N} = 0$$

3. DISCUSSION AND CONCLUDING COMMENTS

The origin of the duality demonstrated above can be traced back to the similar forms of the compatibility eqns (4) and (5) on the one hand and the equilibrium eqns (8) and (9) on the other. Apart from the essential requirement of invertibility, the constitutive relations do not play a direct role in the establishment of the duality. With this point in mind we may examine several special cases by examining the alterations required in the compatibility and equilibrium equations in each case. Consider for instance the case when rotary inertia is neglected. In that case the components Σ_x and Σ_y in equilibrium eqn (9) will vanish. For the dual beam we note that the corresponding quantities in the compatibility equations, namely the shear deformation $(\epsilon_{zx} - \theta_y)$ and $(\epsilon_{zy} + \theta_x)$ must also vanish. A further specialisation is the case of axially rigid rods. In this case $\epsilon_{zz} = 0$ and the modification of the compatibility eqn (1) takes the form of $(\partial u_z/\partial s) = 0$ or u_z = constant along the length of the rod. The dual modification in the equilibrium eqn (7) is that $(\partial M_z/\partial s) = 0$ which can be interpreted as the case of constant axial torque along the rod.

Up till now we have been concerned with situations where work done terms possess appropriate potentials and in eqns (16) and (26) there was no need for separate virtual work terms for prescribed forces and prescribed displacements respectively.

In the complementary energy formulation prescribed forces enter the kinetic energy terms via the equilibrium eqns (8) and (9). In this way the prescribed forces contribute to the momenta of the elements of the rod. In this case the functional (26) does not change, apart from the changes in the expression for *T*. Thus the formulation still remains potential in the sense that separate virtual work terms will not be required for inclusion of the prescribed forces. This is clearly not the case in the Hamiltonian formulation where one must write separate virtual work terms for arbitrarily prescribed forces.

In search of finding a "dual situation" we note that if the rod is subjected to some displacement configuration, then such prescribed displacements will enter the strain energy expression via the compatibility eqns (4) and (5). In this way the prescribed displacements will contribute to the strain distribution in the rod. In this case the functional (16) will not change, except for changes in the expression for U. Thus in the Hamiltonian formulation the problem remains potential in nature. In the complementary formulation however one must account for work of the prescribed displacements by separate virtual work terms [1–3]. From the foregoing we can conclude that a duality can also be established between prescribed forces and prescribed displacements.

Having established a formal duality in the equations of motion of arbitrary shaped rods let us now consider the case of free vibrations. If we denote a natural frequency of the rod by ω then we may write the Rayleigh's quotient in the Hamiltonian formulation as

$$\omega^2 = \frac{\int_0^L \{[\boldsymbol{\theta}' + \boldsymbol{\kappa}_0 \times \boldsymbol{\theta}]\underline{\mathbf{EI}}[\boldsymbol{\theta}' + \boldsymbol{\kappa}_0 \times \boldsymbol{\theta}] + [\mathbf{u}' + \boldsymbol{\kappa}_0 \times \mathbf{u} + \mathbf{t} \times \boldsymbol{\theta}]\underline{\mathbf{GA}} \cdot [\mathbf{u}' + \boldsymbol{\kappa}_0 \times \mathbf{u} + \mathbf{t} \times \boldsymbol{\theta}]\}\,\mathrm{d}s}{\int_0^L (\mathbf{u}\underline{\mathbf{N}}\mathbf{u} + \boldsymbol{\theta}\underline{\mathbf{J}}\boldsymbol{\theta})\,\mathrm{d}s} \tag{34}$$

and in the complementary formulation

$$\omega^2 = \frac{\int_0^L \{[\mathbf{q}' + \boldsymbol{\kappa}_0 \times \mathbf{q}]\underline{\mathbf{N}}^{-1}[\mathbf{q}' + \boldsymbol{\kappa}_0 \times \mathbf{q}] + [\mathbf{m}' + \boldsymbol{\kappa}_0 \times \mathbf{m} + \mathbf{t} \times \mathbf{q}]\underline{\mathbf{J}}^{-1} \cdot [\mathbf{m}' + \boldsymbol{\kappa}_0 \times \mathbf{m} + \mathbf{t} \times \mathbf{q}]\}\,\mathrm{d}s}{\int_0^L (\mathbf{m}\underline{\mathbf{EI}}^{-1}\mathbf{m} + \mathbf{q}\underline{\mathbf{GA}}^{-1}\mathbf{q})\,\mathrm{d}s} \tag{35}$$

Comparison of these quotients shows that providing the correspondence between the dyadics

$$\underline{\mathbf{N}}^{-1} \leftrightarrow \underline{\mathbf{EI}}, \quad \underline{\mathbf{J}}^{-1} \leftrightarrow \underline{\mathbf{GA}},$$

is observed, then the natural frequencies of a system and its dual will be identical. While the mathematical form of the displacement modes $\mathbf{u}$ and $\boldsymbol{\theta}$ will also be identical to the impulse modes $\mathbf{m}$ and $\mathbf{q}$ of the dual system, it must be emphasized that the physical meanings of these modes will not generally be the same. For instance the displacements must satisfy kinematic boundary conditions while the impulses are required to satisfy the force boundary conditions.

4. DUALITIES IN SPECIAL CASES

Next let us examine some special forms of curved and twisted rods. Consider first the equations of a plane arch oscillating in the xz plane. In this case some kinematical and some force quantities vanish. These are

$$\kappa_{z_0} = \kappa_{x_0} = 0 \tag{36}$$

$$q_y = m_z = m_x = 0, \tag{37}$$

The equilibrium and compatibility equations in this case simplify to the following forms respectively.

$$\begin{aligned} q_x' + \kappa_{y0} q_z &= R_x \\ q_z' - \kappa_{y0} q_x &= R_z \\ m_y' + q_x &= \Sigma_y \end{aligned} \tag{38}$$

$$\begin{aligned} u_x' + \kappa_{y0} u_z - \theta_y &= e_x \\ u_z' - \kappa_{y0} u_x &= e_z \\ \theta_y' &= \Delta\kappa_y. \end{aligned} \tag{39}$$

To determine the dual system we first note from eqns (34) and (35) that in this case again

$$\kappa_{z_0} = \kappa_{x_0} = 0. \tag{40}$$

The dual force and kinematic quantities may also be obtained from eqns (34) and (35) or more conveniently from Table 1. Now corresponding to eqn (37) we must require that for the duel system

$$\theta_y = u_z = u_y = 0. \tag{41}$$

Then it can be seen that the *form* of the equilibrium equations of the plane arch in the xz plane, will become identical to the form of the compatibility equations of the dual system and vice versa. Thus for the dual system we have the following equilibrium and compatibility equations respectively.

$$\begin{aligned} m'_x + \kappa_{y_0} m_z - q_y &= \Sigma_x \\ m'_z - \kappa_{y_0} m_x &= \Sigma_z \\ q'_y &= R_y \end{aligned} \qquad \text{(Equilibrium)} \tag{42}$$

$$\begin{aligned} \theta'_x + \kappa_{y_0}\theta_z &= \Delta\kappa_x \\ \theta'_z - \kappa_{y_0}\theta_x &= \Delta\kappa_z \\ u'_y + \theta_x &= e_y. \end{aligned} \qquad \text{(Compatibility)} \tag{43}$$

Table 1. Duality relationships in rods

Inplane			Out of plane		
u_x		u_z	m_x		m_z
	θ_y			q_y	
q_x		q_z	θ_x		θ_z
	m_y			u_y	
R_x		R_z	$\Delta\kappa_x$		$\Delta\kappa_z$
	Σ_y			e_y	
e_x		e_z	Σ_x		Σ_z
	$\Delta\kappa_y$			R_y	

It is now evident that the dual system is the same plane arch but oscillating out of its plane. We should again note that the free and clamped boundary conditions are dual to each other.

Consider finally the case of the straight and pretwisted bars. In this case initial curvatures κ_{x_0} and κ_{y_0} vanish but the twist κ_{z_0}, remains. The equilibrium and compatibility equations now take the following forms, respectively.

$$\begin{aligned} m'_x - \kappa_{z_0} m_y - q_y &= \Sigma_x \\ m'_y + \kappa_{z_0} m_x + q_x &= \Sigma_y \\ M'_z &= \Sigma_z \\ q'_x - \kappa_{z_0} q_y &= R_x \\ q'_y + \kappa_{z_0} q_x &= R_y \\ q'_z &= R_z \end{aligned} \qquad \text{(Equilibrium)}$$

$$\begin{aligned} \theta'_x - \kappa_{z_0}\theta_y &= \Delta\kappa_x \\ \theta'_y + \kappa_{z_0}\theta_x &= \Delta\kappa_y \\ \theta'_z &= \Delta\kappa_z \end{aligned}$$

(Compatibility)

$$u'_x - \kappa_{z_0} u_y - \theta_y = e_x$$

$$u'_y + \kappa_{z_0} u_x + \theta_x = e_y$$

$$u'_z = e_z.$$

In this case the bending and torsional oscillations decouple—as expected. Also in the present case it is not necessary to seek the dual system for it can be seen that the equilibrium equations of the pretwisted rod are directly dual to the compatibility equations of the same rod! Hence pretwisted rods are self dual. Thus, for instance, a uniform pretwisted rod will have natural frequencies which are identical to those of the same rod with its ends free. This is a generalisation of a result which is well known for uniform prismatic rods [4, 5].

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